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Evolution of high Reynolds number two-dimensional turbulence

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Kraichnan's (1967) predictions concerning a simultaneous direct enstrophy cascade and inverse energy cascade for high Reynolds number two-dimensional turbulence are tested numerically using a variant of the eddy-damped quasi-normal approximation. For the initial-value problem, an analytic study using this theory shows that, in the zero-viscosity limit, energy and enstrophy are conserved for arbitrarily long times, contrary to the three-dimensional case, where the energy is conserved for only a finite time, after which it is dissipated. Non-local effects in the enstrophy inertial range, which are difficult to treat by conventional numerical schemes (Leith 1971; Leith & Kraichnan 1972), are shown to be representable by an additional diffusion term in the spectral equation. The resulting equation, including non-local effects, is integrated numerically. When enstrophy and energy are continuously injected at a fixed wavenumber, it is shown numerically that a quasi-steady regime is obtained where enstrophy cascades to large wavenumbers across a k^{-3} inertial range with zero energy transfer while energy flows indefinitely to small wavenumbers across a $k^{-\frac{5}{3}}$ inertial range with zero enstrophy transfer.

1. Introduction

The essential feature of two-dimensional incompressible turbulence as opposed to three-dimensional turbulence is the conservation of the total enstrophy $D(t) = \langle(\operatorname{curl}\mathbf{u})^2\rangle$ for differentiable solutions of the inviscid Navier–Stokes equations (Euler equations). As a consequence, it is not possible to generalize the Kolmogorov concept of an energy cascade from energy-containing eddies to small

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eddies, since such a cascade would increase the enstrophy. Nevertheless, owing to the conservation of enstrophy, the existence of an enstrophy cascade has been conjectured, where the energy spectrum $E(k,t)$ follows a k^{-3} law (Kraichnan 1967; Leith 1968; Batchelor 1969) possibly with a logarithmic correction (Kraichnan 1971*b*). Kraichnan's (1967) complete hypothesis assumes also the existence of an inverse $k^{-\frac{5}{3}}$ energy cascade from energy-containing eddies to larger eddies. Enstrophy transfer towards large wavenumbers would then increase the 'palinstrophy'†

$$P(t) = \langle (\nabla^2 \mathbf{u})^2 \rangle = \langle \{\mathrm{curl}\,(\mathrm{curl}\,\mathbf{u})\}^2 \rangle = \int_0^\infty k^4 E(k,t)\, dk, \tag{1.1}$$

where $E(k,t)$ is defined by

$$\langle u^2(t) \rangle = \int_0^\infty E(k,t)\, dk.$$

The conjecture of an enstrophy cascade appears to be relevant to atmospheric measurements (Wiin-Nielsen 1967; Morel & Necco 1973; Desbois 1975), since large-scale atmospheric motions are known to be quasi-two-dimensional. On the other hand, direct numerical simulations of two-dimensional turbulence carried out by Lilly (1969) and by Herring *et al.* (1974) seem to be consistent with the existence of an inverse energy cascade. However, the Reynolds numbers considered in these numerical simulations are rather small (< 1000). Such Reynolds numbers are certainly too low to produce a clear-cut enstrophy inertial range.

In order to study two-dimensional turbulence at large Reynolds numbers, and particularly to decide whether an enstrophy cascade exists or not, it is convenient to make use of the stochastic models introduced by Kraichnan (1958, 1961). The stochastic models are chosen so as to embody most of the structural properties of the Navier–Stokes equations and to lead to closed master equations for mean quantities. Since the existence of energy and enstrophy cascades seems to be closely related to the conservation of energy and enstrophy by the nonlinear terms of the Navier–Stokes equations and to the random Galilean invariance of the equations, a model having these properties seems most suitable. In this paper, we shall work with a Markovian eddy-damped model of the class of those introduced by Leith (1971) and Herring & Kraichnan (1972). Such models are characterized by a time $\theta_{kpq}(t)$ for the relaxation of triple correlations (Kraichnan 1971*a*; Frisch, Lesieur & Sulem 1974). Here we shall take for $\theta_{kpq}(t)$ the following expression:

$$\theta_{kpq}(t) = t/[1 + (\mu_k + \mu_p + \mu_q)\, t], \tag{1.2}$$

with the eddy-damping rate μ_k given by

$$\mu_k = \nu k^2 + \lambda \left(\int_0^k p^2 E(p,t)\, dp \right)^{\frac{1}{2}}, \tag{1.3}$$

where λ is a constant which will be chosen to fit given values of Kolmogorov constants. In (1.2), θ_{kpq} is chosen in such a way that $\theta_{kpq} = t$ for small times and

† From the Greek παλιν (again) and στρωγη (rotation, curl).

$\theta_{kpq} = (\mu_k + \mu_p + \mu_q)^{-1}$ for large times.† Once an inertial range has been established a more local expression of the form $\mu_k = \nu k^2 + \lambda'\{k^3 E(k,t)\}^{\frac{1}{2}}$ (Orszag 1970) would be as convenient; such an expression is however totally inadequate for short times and initially rapidly decreasing spectra: indeed μ_k would then be an exponentially decreasing function of k for moderately large wavenumbers and hence the relaxation time for triple correlations would be larger than the large eddy turnover time. Our expression has the advantage that it takes into account the strong non-localness of the enstrophy transfer, since it is known that in two-dimensional turbulence wavenumbers much smaller than k contribute substantially to the shearing of wavenumbers $\sim k$ (Kraichnan 1971*b*). Also, it can be shown (Herring, private communication) that expression (1.3) for the eddy-damping rate can be considered as an approximation to the eddy-damping rate given by the test-field model (Kraichnan 1971*a*).

It must be mentioned that several two-dimensional predictability studies, e.g. the study of the propagation of a perturbation in a given energy spectrum, have been made using stochastic models (Leith 1971; Leith & Kraichnan 1972; Herring 1973). But the evolution of a given energy spectrum with or without external forces has never been computed for high Reynolds numbers, so that dynamical evidence is still lacking for the existence of the energy and enstrophy cascades. The methods used here are new in that they combine an accurate treatment of distant, diffusive interactions with strict energy and enstrophy conservation, all of which are necessary to obtain reliable large Reynolds number results.

2. The absence of an enstrophy catastrophe

The eddy-damped master equation for the energy spectrum $E(k,t)$ reads in the case of two-dimensional isotropic turbulence (Leith 1971)

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k,t) = \frac{1}{\pi}\iint_{\Delta_k} \frac{k^2}{pq}\,\theta_{kpq}(t)\, b_2(k,p,q) \times \{kE(p,t)\,E(q,t) - pE(q,t)\,E(k,t)\}\,dp\,dq + F(k), \tag{2.1}$$

where the integration in the p, q plane extends over the domain Δ_k such that $\mathbf{k}$, $\mathbf{p}$ and $\mathbf{q}$ can be the three sides of a triangle. The coefficient $b_2(k,p,q)$ is given by

$$b_2(k,p,q) = 2\frac{p}{k}\frac{(xy - z + 2z^3)}{(1-x^2)^{\frac{1}{2}}} = 2\frac{(k^2-q^2)(p^2-q^2)(1-x^2)^{\frac{1}{2}}}{k^4}, \tag{2.2}$$

where x, y and z are the cosines of the interior angles of the (k,p,q) triangle. $F(k)$ is a forcing term which injects energy at a wavenumber k_I. In this section $F(k)$ will be taken equal to zero.

It has been conjectured for three-dimensional isotropic turbulence that the existence of an energy cascade is the consequence of an energy catastrophe in the zero-viscosity limit (Onsager 1949; Proudman & Reid 1954; Brissaud *et al.* 1973),

† Notice that Leith's (1971) expression $\theta_{kpq}(t) = [1 - \exp\{-(\mu_k + \mu_p + \mu_q)t\}]/(\mu_k + \mu_p + \mu_q)$ for θ_{kpq} is not substantially different. We have chosen (1.2) to save computing time.

i.e. inviscid dissipation of kinetic energy:

$$\lim_{\nu\to 0} \epsilon(t) = \lim_{\nu\to 0} 2\nu \int_0^\infty k^2 E(k,t)\, dk \neq 0, \tag{2.3}$$

where $\epsilon(t)$ is the energy viscous dissipation rate. In three-dimensional turbulence, there is a finite rate of energy dissipation by an infinitesimal viscosity because the energy cascade process makes the enstrophy $D(t)$ increase towards infinity in such a way that the product

$$2\nu \int_0^\infty k^2 E(k,t)\, dk$$

remains different from zero in the limit $\nu \to 0$. Furthermore, Orszag (1974) noted that the three-dimensional energy catastrophe would occur after a finite time, at which time the enstrophy would become infinite. This conjecture has been analytically and numerically verified (Brissaud *et al.* 1973) using the three-dimensional Markovian random coupling model, where θ_{kpq} is set equal to a constant (Frisch, Lesieur & Brissaud 1974).

For two-dimensional turbulence, there is obviously no possibility of an energy catastrophe, since the constraint of enstrophy conservation imposes in (2.3) a zero limit on $\mathscr{E}(t)$ when $\nu \to 0$. But from the hypothesis of an enstrophy cascade one is quite naturally led to examine the possibility of an enstrophy catastrophe (Batchelor 1969):

$$\lim_{\nu\to 0} \beta(t) = \lim_{\nu\to 0} 2\nu \int_0^\infty k^4 E(k,t)\, dk \neq 0. \tag{2.4}$$

However, we shall show that within the framework of the eddy-damped quasi-normal approximation the limit (2.4) remains zero for arbitrarily large times. This result, conjectured on a phenomenological basis by Orszag (1974), can be demonstrated analytically without any restriction on $\theta_{kpq}(t)$ other than that it should be bounded by t. In particular, this result holds for the test-field model (Kraichnan 1971*a*). Indeed the equation for the rate of change of the palinstrophy (we put the forcing equal to zero, which in the present case is irrelevant) reads

$$\frac{d}{dt} P(t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty p^4 q^2 E(p,t)\, E(q,t)\, B(p,q)\, dp\, dq, \tag{2.5}$$

where $B(p,q)$ is given by

$$B(p,q) = \left(1 - \frac{q^2}{p^2}\right) \int_{-1}^{+1} \frac{(1-x^2)^{\frac{1}{2}}}{1 - 2qx/p + q^2/p^2}\, \theta_{kpq}(t)\, dx. \tag{2.6}$$

Equations (2.5) and (2.6) can be established following the lines of the calculation made by Lilly (1971) in the case of the quasi-normal theory. In (2.6), as the integrand is always positive and $\theta_{kpq}(t)$ is bounded by t, we have

$$B(p,q) \leqslant t \int_{-1}^{+1} \frac{(1-x^2)^{\frac{1}{2}}}{1 - 2qx/p + q^2/p^2}\, dx. \tag{2.7}$$

The integral in (2.7) is equal to $\frac{1}{2}\pi$ for $q/p \leqslant 1$ and $\frac{1}{2}\pi p^2/q^2$ for $q/p > 1$, so that

$B(p,q)$ can be further bounded by

$$B(p,q) \leqslant \tfrac{1}{2}\pi t. \tag{2.8}$$

Inserting (2.8) in (2.5) we finally get

$$dP(t)/dt \leqslant tP(t)\,D(t). \tag{2.9}$$

Since $D(t) \leqslant D(0)$, we can write

$$P(t) \leqslant P(0)\exp\{\tfrac{1}{2}D(0)\,t^2\}, \tag{2.10}$$

which shows that at any finite time the palinstrophy remains finite and hence $\lim_{\nu\to 0} \beta(t) = 0$. There is no enstrophy catastrophe.

In addition to this calculation, an argument can be given to suggest that there is no *need* for an enstrophy catastrophe occurring after a finite time. In the steady regime, when there is a constant enstrophy injection rate, the enstrophy spectrum is $\sim k^{-1}$ towards large k, possibly with a logarithmic correction (see next section); in any case, the total enstrophy diverges for large k, so that it is not necessary to extract enstrophy by a catastrophe to maintain steadiness. Notice also that, if this no-catastrophe result is true for the original two-dimensional Euler equations,† then it probably implies that intermittency cannot steepen the power law of the energy spectrum in the enstrophy cascade, since a steepening of the energy spectrum by intermittency would make the total enstrophy converge for large k, thus requiring a catastrophe to extract the injected enstrophy (Frisch, private communication).

3. The non-localness of enstrophy transfer

The numerical method used for the calculation of double integrals over wavenumbers in (2.1) is essentially the same as that in Leith (1971). We take a logarithmic subdivision of the k axis

$$k_L \sim 2^{L/F}. \tag{3.1}$$

Such a logarithmic subdivision has the consequence that, because isosceles interactions ($k = q, p = q, k = p$) in the model equation are zero for two-dimensional turbulence, the numerical calculation of double integrals in the p, q plane cuts off all the 'non-local' interactions such that the ratio of the smallest to the middle wavenumber in the interacting triad (k,p,q) is less than $a = 2^{1/F}-1$ (equal to 0·19 when $F = 4$). We know from the work of Kraichnan (1971 *b*), however, that non-local interactions are responsible for most of the enstrophy transfer.

The first method to take into account non-local interactions was proposed by Leith & Kraichnan (1972), who used a much more refined discretization of wavenumbers for small values of p/k and q/k in the interacting triad. Here we introduce a method based on the expansion of the nonlinear term of the spectral equation (2.1), which is now rewritten as

$$(\partial/\partial t + 2\nu k^2)\,E(k,t) - F(k) = T(k,t) = T_{\rm L}(k,t) + T_{\rm NL}(k,t), \tag{3.2}$$

† This result has been established by Bardos & Frisch (1974) for the spatially periodic case.

where T_{L} and T_{NL} stand for the local and non-local contribution to the transfer function. There are two classes of non-local interactions:

(i) $q \ll p \sim k, \quad p \ll q \sim k,$

(ii) $k \ll p \sim q.$

As shown by Kraichnan (1971*b*), the dominant contribution comes from the first class; a suitable expansion in powers of a then gives to leading order

$$T_{\mathrm{NL}}(k,t) = \frac{1}{8}\frac{1}{k^2}\frac{\partial}{\partial k}\left\{\left[k^3\int_0^{ak}\theta_{kkq}(t)\,q^2E(q)\,dq\right]\frac{\partial}{\partial k}kE(k)\right\}. \tag{3.3}$$

At this point a problem arises with energy conservation. It is essential to have a numerical scheme which conserves exactly both energy and enstrophy (otherwise cascade phenomena are easily lost, as, for example, in Dupree 1974). The local term T_{L} has exact conservation properties because exact conservation holds for each triad (Kraichnan 1967); this property is preserved in Leith's numerical scheme (within rounding errors). The non-local term T_{NL}, however, does not have all the right conservation properties. This is because detailed conservation requires the inclusion of terms which are not of the same order in a. The fact that $T_{\mathrm{NL}}(k,t)$ nevertheless has exact enstrophy conservation [easily checked in (3.3)] is due to pathological behaviour of two-dimensional turbulent viscosities, which are known to be zero to leading order (Kraichnan 1975). Exact energy conservation may be achieved without including explicitly class (ii) interactions by replacing T_{NL} with

$$T'_{\mathrm{NL}}(k,t) = \frac{1}{8}\frac{1}{k^2}\frac{\partial}{\partial k}\left\{k^3\frac{\partial}{\partial k}\left[kE(k)\int_0^{ak}\theta_{kkq}(t)\,q^2E(q)\,dq\right]\right\}, \tag{3.4}$$

which, because of the logarithmic behaviour of the integral in the enstrophy cascade, introduces a relative error of order $(\log k)^{-1}$. After these remarks, we are led to use instead of (3.2) the equation

$$(\partial/\partial t + 2\nu k^2)\,E(k,t) - F(k) = T_{\mathrm{L}} + T'_{\mathrm{NL}}. \tag{3.5}$$

It can be checked analytically that in the limit of infinite Reynolds number (3.5) has steady solutions of the form given by Kraichnan (1971*b*):

$$E(k) = 4{\cdot}19\lambda^{\frac{2}{3}}\beta^{\frac{2}{3}}k^{-3}\{\log(k/k_1)\}^{-\frac{1}{3}} \quad \text{for} \quad k \gg k_I, \tag{3.6}$$

with
$$\beta = \int_0^\infty k^2F(k)\,dk.$$

For such solutions the local enstrophy transfer is negligible. It is not entirely clear how k_1 should be chosen in (3.6). There is no reason why k_1 should be equal to the injection wavenumber k_I since the form (3.6) of the spectrum holds only for $k \gg k_I$. In principle k_1 can be determined by fitting (3.6) to the result of a high Reynolds number numerical calculation but this gives rather poor accuracy unless extremely high Reynolds numbers (of the order of 10^{20}) are used. There are indications that k_0 is smaller than k_I by a factor of 10^2–10^3. As for the adjustable constant λ appearing in (1.3), we have taken the value 0·376 by noting that the enstrophy inertial range has a Kolmogorov constant given by

$$C' = 4{\cdot}19\lambda^{\frac{2}{3}}, \tag{3.7}$$

which we have fitted to the value derived by Leith & Kraichnan (1972) from the test-field model.†

4. Numerical results

We carried out a numerical integration of (3.5) both for the free evolution of a given initial energy spectrum and for the case of turbulence driven by external forces. The numerical scheme used for T'_{NL} is described in the appendix. In all the following calculations we take $F = 4$.

Time evolution of an initial state

We integrated (3.5) numerically with $F(k) = 0$ and the initial condition

$$E(k, 0) \sim k^3 \exp -\{\tfrac{3}{2}(k/k_I)^2\}. \tag{4.1}$$

In order to estimate approximately the viscous cut-off wavenumber k_D in the numerical integration, we proceed as follows: assuming a k^{-3} inertial range extending from k_I to k_D in the energy spectrum, we can write the enstrophy dissipation rate $\beta(t)$ as

$$\beta(t) = 2\nu \int_0^\infty k^4 E(k,t)\, dk \sim \nu \int_0^\infty k^4 (\beta(t))^{\frac{2}{3}} k^{-3} dk \sim (\beta(t))^{\frac{2}{3}} k_D^2 \nu. \tag{4.2}$$

On the other hand, if the viscosity is small enough the enstrophy $D(t)$ is not far below its initial value $D(0)$, so that we have

$$D(0) \simeq \int_{k_I}^{k_D} k^2 E(k)\, dk \sim \int_{k_I}^{k_D} (\beta(t))^{\frac{2}{3}} k^{-1} dk = (\beta(t))^{\frac{2}{3}} \log\left(\frac{k_D}{k_I}\right). \tag{4.3}$$

Eliminating $\beta(t)$ between relations (4.2) and (4.3) we find

$$D(0) \sim k_D^4 \nu^2 \log(k_D/k_I). \tag{4.4}$$

The logarithmic factor $\log(k_D/k_I)$ is of order one and the initial enstrophy $D(0)$ is of order $k_I^2\langle u^2(0)\rangle$, so that (4.4) becomes

$$(k_D/k_I)^2 \sim (\langle u^2(0)\rangle)^{\frac{1}{2}}/\nu k_I = R, \tag{4.5}$$

where R is the large-scale Reynolds number.

The results of numerical integration of (3.5) without forcing at a Reynolds number $R = 2{\cdot}4 \times 10^7$ are presented in figure 1. The units are k_I for k,

$$\{(\langle u^2(0)\rangle)^{\frac{1}{2}} k_I\}^{-1}$$

for t and $\langle u^2(0)\rangle/k_I$ for $E(k,t)$. Log $E(k, t)$ is plotted *vs.* $\log k$ for times $t = 0$, 400 and 1200.

† With our definition of the energy spectrum,

$$\langle u^2\rangle = \int_0^\infty E(k)\, dk,$$

our Kolmogorov constant must be multiplied by $2^{-\frac{1}{3}}$ to get the value of 1·74 given by Leith & Kraichnan (1972).

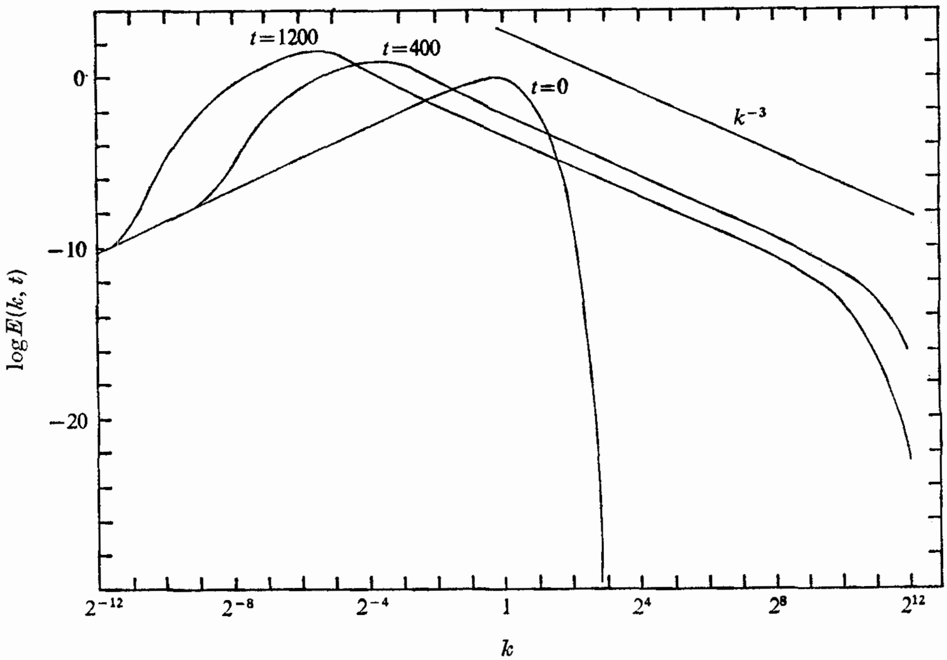

FIGURE 1. Free evolution of the energy spectrum without forcing. Initial spectrum $E(k,0) \sim k^3 \exp\{-\frac{3}{2}(k/k_I)^2\}$. Reynolds number $R = 2{\cdot}4 \times 10^7$.

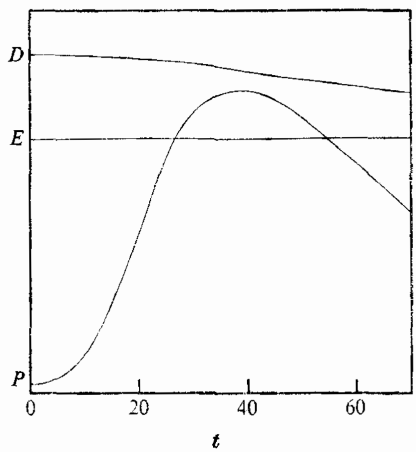

FIGURE 2. Variation of total energy $E = \langle u^2 \rangle$, total enstrophy $D = \langle (\text{curl}\,\mathbf{u})^2 \rangle$ and total palinstrophy $P = \langle (\text{curl curl}\,\mathbf{u})^2 \rangle$ in arbitrary units for two-dimensional turbulence with the same initial conditions as in figure 1.

We first notice that there is a backward flux of energy towards low wave-numbers. Second, we see clearly the formation of a k^{-3} enstrophy-cascading inertial range extending for $t = 1200$ from $k = 2^{-5}$ to $k = 2^9$. It must be emphasized that in the limit of zero viscosity we cannot expect a k^{-3} enstrophy cascade (even corrected by a logarithmic factor) extending to infinity: indeed this would imply an infinite total enstrophy, which is not possible in the unforced case since enstrophy is bounded above by its initial value.

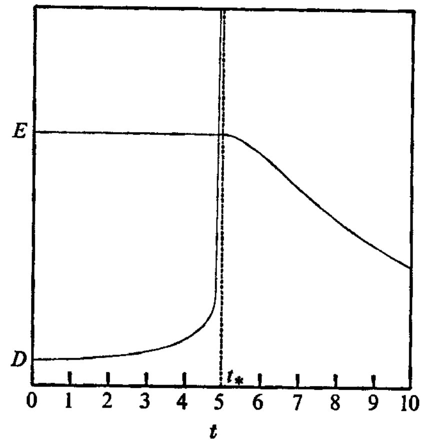

FIGURE 3. Variation of total energy and total enstrophy for three-dimensional turbulence in the limit of infinite Reynolds number. Notice the onset of 'inviscid dissipation' after t_* (taken from Lesieur 1973).

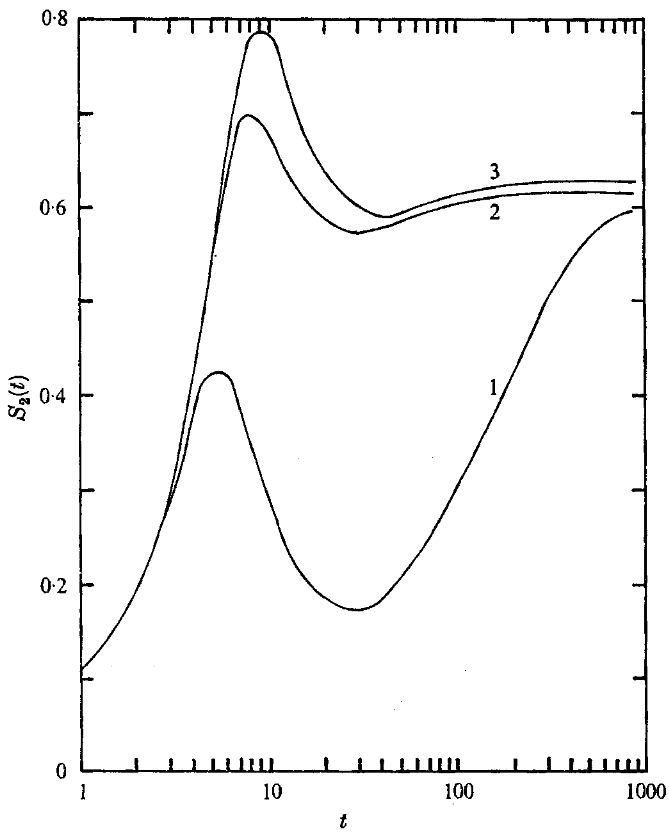

FIGURE 4. Variation of the skewness factor as a function of time for different values of the Reynolds number. Curve 1, $R = 10^4$; curve 2, $R = 2{\cdot}6\times10^5$; curve 3, $R = 2{\cdot}4\times10^7$.

Figure 2 shows the variation of the total energy, enstrophy and palinstrophy; as expected from the results of § 2, energy and enstrophy remain practically constant while the palinstrophy has an exponential growth limited only by viscosity. For comparison, figure 3 shows typical behaviour of the energy and enstrophy for three-dimensional inviscid turbulence (Lesieur 1973).

Figure 4 shows the two-dimensional skewness factor, as defined by Herring *et al.* (1974),

$$S_2(t) = \int_0^\infty k^4 T(k,t)\, dk / P(t)\, D(t)^{\frac{1}{2}}, \tag{4.6}$$

where $T(k,t)$ is the right-hand side of (3.5), for $R = 10^4$, $2{\cdot}6 \times 10^5$ and $2{\cdot}4 \times 10^7$. One sees that, at $t = 1000$, $S_2(t)$ has reached a steady value close to 0·6, which increases slowly with Reynolds number. Notice that this value† is about twice the low Reynolds number skewness factors computed by Herring *et al.* (1974).

Turbulence driven by external forces

In the preceding case, it was not possible to obtain an inverse $k^{-\frac{5}{3}}$ energy inertial range because there was no energy input into the system. Now we are going to consider the case where there is a forcing term $F(k)$ in (3.5), whose effect is to inject energy and enstrophy in a narrow band in the neighbourhood of the wavenumber k_I at given rates ϵ and $\beta = k_I^2 \epsilon$. From preceding results on free turbulence and phenomenological predictions, one may foresee that enstrophy will cascade to large k and that the part of the spectrum corresponding to $k > k_I$ will quickly become steady with an enstrophy dissipation rate equal to the injection rate β. Energy, which cannot be significantly transferred towards large k, will then cascade backwards according to a $k^{-\frac{5}{3}}$ law. At time t, the total energy will be $\epsilon t + \langle u^2(0) \rangle$, and the excitation will have reached a wavenumber $k_{\min}(t)$ approximately given by

$$\epsilon t + \langle u^2(0) \rangle \sim \int_{k_{\min}(t)}^{k_I} \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} dk. \tag{4.7}$$

For large t, we have $\langle u^2(0) \rangle \ll \epsilon t$ and $k_I \gg k_{\min}(t)$, so that (4.7) becomes

$$k_{\min}(t) \sim (\epsilon t^3)^{-\frac{1}{2}}. \tag{4.8}$$

Figure 5 shows the temporal evolution of the spectrum $E(k,t)$ for $R = 2{\cdot}4 \times 10^7$ at times $t = 0, 100, 1000$ and 3000 (the computing time was 1400 s on a CDC 7600 machine). We see very clearly that a steady k^{-3} enstrophy cascade extending from $k = 2^2$ to $k = 2^{10}$ is rapidly established, and that energy goes back towards small k across a $k^{\frac{5}{3}}$ inverse cascade. At $t = 3000$, the $k^{-\frac{5}{3}}$ cascade extends from $k = 2^{-1}$ to $k = 2^{-7}$. The corresponding energy Kolmogorov constant is found to be 10 % less than the value determined by Kraichnan (1971*b*) from the test-field model. This shows that the model used here gives slightly different results from the test-field model and that it is not possible to adjust λ to recover exactly both Kolmogorov constants of the test-field model. If we integrated for longer

† To be consistent with the notation of Herring *et al.* (1974) we must multiply our skewness factor by $2^{\frac{1}{2}}$, which gives a value of 0·85.

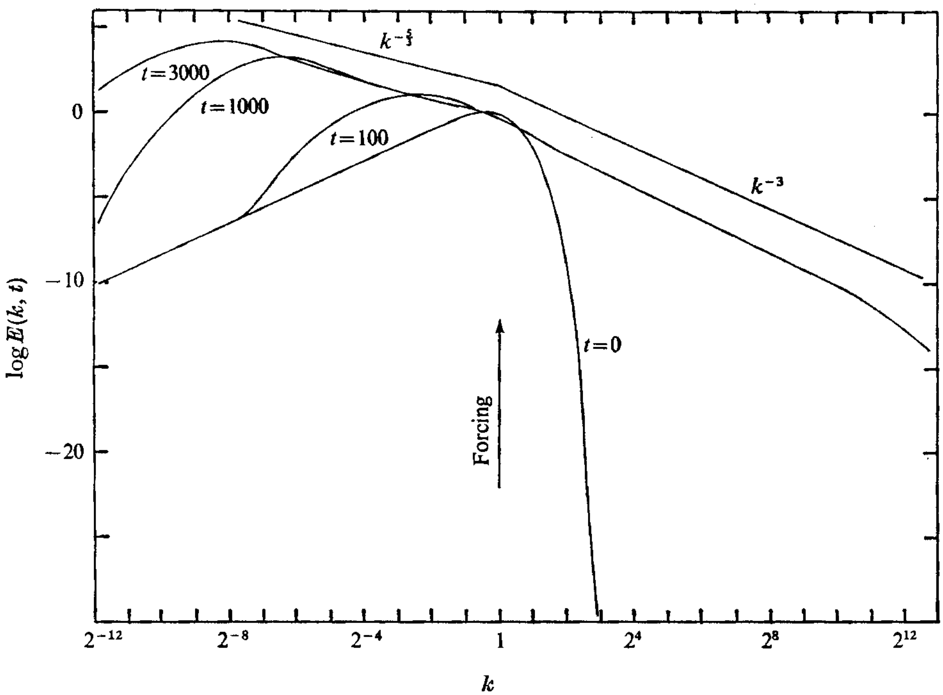

FIGURE 5. Quasi-steady energy spectrum $E(k, t)$ for $t = 100$, 1000 and 3000 corresponding to an injection spectrum constant in a half-octave band around $k_I = 1$ with injection rates $\epsilon = 0{\cdot}03$ and $\beta = 0{\cdot}03$. Reynolds number $R = 2{\cdot}4 \times 10^7$.

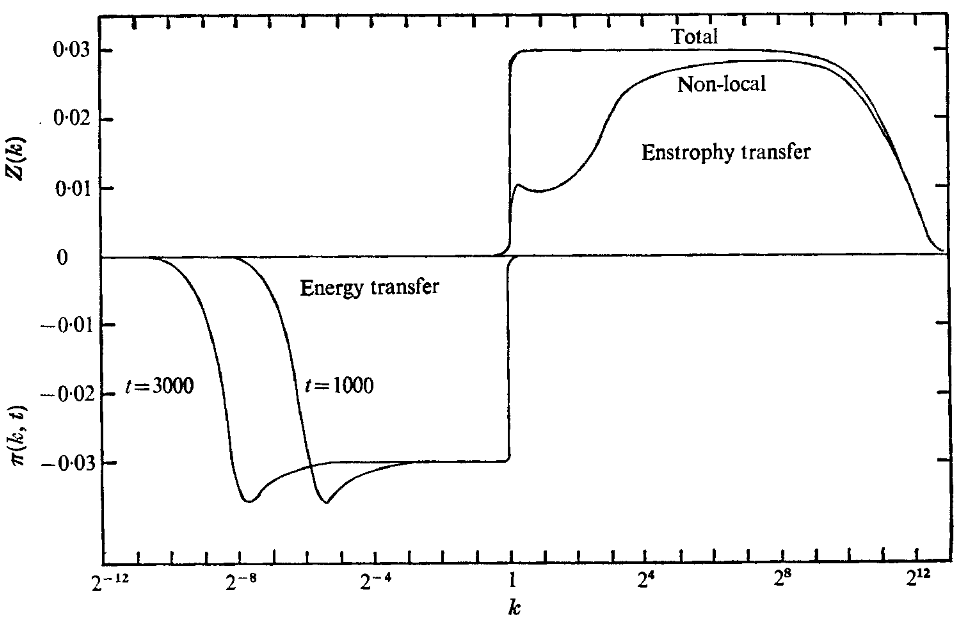

FIGURE 6. Quasi-steady energy transfer rate $\Pi(k, t)$ and enstrophy transfer rates $Z(k, t)$ and $Z_{NL}(k, t)$ for $t = 1000$ and 3000; same conditions as in figure 5.

times, the numerical results would no longer be significant for small k, since energy would accumulate on the lower k cut-off k_{low} instead of feeding progressively the infinite space left under the $k^{-\frac{5}{3}}$ spectrum. In that case ($k_{\text{low}} \neq 0$) this accumulation does not continue for ever and the total energy is easily shown to be bounded by noticing that its time derivative is less than $\epsilon - 2\nu k_{\text{low}}^2 \langle u^2(t)\rangle$ and integrating.

Figure 6 shows for times $t = 1000$ and 3000 the transfer rate of the energy,

$$\Pi(k,t) = \int_k^\infty T(p,t)\,dp, \tag{4.9}$$

the transfer rate of the enstrophy,

$$Z(k,t) = \int_k^\infty p^2 T(p,t)\,dp, \tag{4.10}$$

and the 'non-local' enstrophy transfer rate

$$Z_{\text{NL}}(k,t) = \int_k^\infty p^2\, T'_{\text{NL}}(p,t)\,dp. \tag{4.11}$$

For $t = 3000$, as expected, the energy transfer rate $\Pi(k,t)$ is negative in the inverse energy cascade and vanishes in the direct enstrophy cascade, while the enstrophy transfer rate vanishes for $k < k_I$ and is positive in the enstrophy cascade; in the inverse energy cascade, $\Pi(k)$ remains constant to within less than 1 % for $2^{-5} \leqslant k \leqslant 1$, whereas in the enstrophy cascade $Z(k)$ remains constant to within less than 1 % for $1 < k \leqslant 2^8$. The sharp discontinuity in the energy and enstrophy transfer at k_I seen in figure 6 is probably due to the forcing term, which is sharply peaked near k_I. One can check that most of the enstrophy transfer comes from non-local interactions. It would be tempting to neglect the local term T_{L} in the spectral equation and keep only the non-local term T'_{NL}, but this is not possible for small values of k ($\lesssim k_I$) since then the non-local term is negligible and does not transfer energy and enstrophy. Furthermore it may be shown (Herring, private communication) that such a spectral equation would lead in the unforced case to a k^{-2} energy spectrum. Notice finally that the skewness factor in this experiment is 0·57 and that the Reynolds number of this numerical calculation is too small to exhibit clearly the logarithmic correction to the k^{-3} range.

5. Conclusion

Kraichnan's hypotheses (direct enstrophy cascade and inverse energy cascade) concerning two-dimensional isotropic turbulence have always been much debated (e.g. Saffman 1971), and neither cascade has been clearly demonstrated, either numerically or experimentally, although Kolesnikov & Tsinober (1972) claim that a flow of mercury imbedded in a sufficiently strong magnetic field becomes two-dimensional and displays both $k^{-\frac{5}{3}}$ and k^{-3} inertial ranges; also, in the atmosphere, where the k^{-3} law seems to fit observations (Wiin-Nielsen 1967; Desbois 1974; Morel & Larchevêque 1974; Morel & Necco 1973), the energy-containing eddies are of the same order of magnitude as large-scale eddies which dissipate energy by friction with the surface layer, and therefore there is not a wide enough range for the inverse cascade to establish itself. On the other hand, the direct numerical simulations of Lilly (1969, 1972) and Herring *et al.* (1974) show a tendency to establish both cascades, but such calculations are severely restricted by the capacity of computers. Since at present direct numerical simulation of turbulence at very high Reynolds number is impossible, it seems that the best way to get insight into the dynamics is to work with stochastic models.

After the work of Kraichnan (1971*b*) and Leith (1971) and the results of the present paper, the existence of the two cascades seems now to be clearly established for the class of Markovian eddy-damped theories: in these models the dynamics of two-dimensional turbulence are fairly well understood. The structural similarity between the stochastic models and the Navier–Stokes equations and the good agreement of test-field-model results with low Reynolds number direct numerical simulations (Herring *et al.* 1974) give strong arguments in favour of Kraichnan's hypothesis. Nevertheless, stochastic models differ from the two-dimensional Navier–Stokes equations in two essential points: they do not conserve the mean values of all powers of the vorticity, contrary to the original Euler equations, and they suppress intermittency. It seems, however, that intermittency does not affect the energy spectrum in the enstrophy cascade (Kraichnan 1975).

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Appendix

Let us write $T'_{\mathrm{NL}}(k)$ in the form

$$T'_{\mathrm{NL}}(k) = \frac{1}{8}\frac{\partial}{\partial k}\left\{\frac{1}{k}\frac{\partial}{\partial k}(k^2\psi(k))\right\}, \tag{A 1}$$

where

$$\psi(k) = kE(k)\int_0^{ak}\theta_{kkq}(t)\,q^2E(q,t)\,dq. \tag{A 2}$$

The differential operator $\partial/\partial k$ is approximated by the finite-difference operator δ:

$$\delta\phi(k_0) = \frac{\phi(\alpha k_0)-\phi(\alpha^{-1}k_0)}{\alpha k_0-\alpha^{-1}k_0}, \tag{A 3}$$

with $\alpha = 2^{1/2F}$. Then the scheme for T'_{NL} reads

$$T'_{\mathrm{NL}}(k_0) = (c/8k_0)\{\alpha^4\psi(\alpha^2k_0)-(\alpha^4+1)\,\psi(k_0)+\psi(\alpha^{-2}k_0)\}, \tag{A 4}$$

where

$$c = \alpha^{-2}(\alpha-\alpha^{-1})^{-2}. \tag{A 5}$$

It can be easily checked that this scheme conserves both energy and enstrophy.

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